Finite-Element Method

To this juncture, we have employed finite-difference methods to solve partial differential equations. In these methods, the solution domain is divided into a grid of discrete points or nodes (Fig. 31.1b). The PDE is then written for each node and its derivatives replaced by finite-divided differences. Although such "pointwise" approximation is conceptually easy to understand, it has a number of shortcomings. In particular, it becomes harder to apply for systems with irregular geometry, unusual boundary conditions, or heterogenous composition.

The finite-element method provides an alternative that is better suited for such systems. In contrast to finite-difference techniques, the finite-element method divides the solution domain into simply shaped regions, or "elements" (Fig. 31.1c). An approximate solution for

**FIGURE 31.1**
(a) A gasket with irregular geometry and nonhomogeneous composition. (b) Such a system is very difficult to model with a finite-difference approach. This is due to the fact that complicated approximations are required at the boundaries of the system and at the boundaries between regions of differing composition. (c) A finite-element discretization is much better suited for such systems.
the PDE can be developed for each of these elements. The total solution is then generated by linking together, or “assembling,” the individual solutions taking care to ensure continuity at the interelement boundaries. Thus, the PDE is satisfied in a piecewise fashion.

As in Fig. 31.1c, the use of elements, rather than a rectangular grid, provides a much better approximation for irregularly shaped systems. Further, values of the unknown can be generated continuously across the entire solution domain rather than at isolated points.

Because a comprehensive description is beyond the scope of this book, this chapter provides a general introduction to the finite-element method. Our primary objective is to make you comfortable with the approach and cognizant of its capabilities. In this spirit, the following section is devoted to a general overview of the steps involved in a typical finite-element solution of a problem. This is followed by a simple example: a steady-state, one-dimensional heated rod. Although this example does not involve PDEs, it allows us to develop and demonstrate major aspects of the finite-element approach unencumbered by complicating factors. We can then discuss some issues involved in employing the finite-element method for PDEs.

### 31.1 THE GENERAL APPROACH

Although the particulars will vary, the implementation of the finite-element approach usually follows a standard step-by-step procedure. The following provides a brief overview of each of these steps. The application of these steps to engineering problem contexts will be developed in subsequent sections.

#### 31.1.1 Discretization

This step involves dividing the solution domain into finite elements. Figure 31.2 provides examples of elements employed in one, two, and three dimensions. The points of intersection of the lines that make up the sides of the elements are referred to as nodes and the sides themselves are called nodal lines or planes.

#### 31.1.2 Element Equations

The next step is to develop equations to approximate the solution for each element. This involves two steps. First, we must choose an appropriate function with unknown coefficients that will be used to approximate the solution. Second, we evaluate the coefficients so that the function approximates the solution in an optimal fashion.

**Choice of Approximation Functions.** Because they are easy to manipulate mathematically, polynomials are often employed for this purpose. For the one-dimensional case, the simplest alternative is a first-order polynomial or straight line,

\[ u(x) = a_0 + a_1 x \]

where \( u(x) \) is the dependent variable, \( a_0 \) and \( a_1 \) are constants, and \( x \) is the independent variable. This function must pass through the values of \( u(x) \) at the end points of the element at \( x_1 \) and \( x_2 \). Therefore,

\[ u_1 = a_0 + a_1 x_1 \]
\[ u_2 = a_0 + a_1 x_2 \]
where \( u_1 = u(x_1) \) and \( u_2 = u(x_2) \). These equations can be solved using Cramer’s rule for

\[
\begin{align*}
a_0 &= \frac{u_1x_2 - u_2x_1}{x_2 - x_1} \\
a_1 &= \frac{u_2 - u_1}{x_2 - x_1}
\end{align*}
\]

These results can then be substituted into Eq. (31.1) which, after collection of terms, can be written as

\[
u = N_1u_1 + N_2u_2
\]

(31.2)

where

\[
N_1 = \frac{x_2 - x}{x_2 - x_1}
\]

(31.3)

and

\[
N_2 = \frac{x - x_1}{x_2 - x_1}
\]

(31.4)

Equation (31.2) is called an approximation, or shape function, and \( N_1 \) and \( N_2 \) are called interpolation functions. Close inspection reveals that Eq. (31.2) is, in fact, the Lagrange
first-order interpolating polynomial. It provides a means to predict intermediate values (that is, to interpolate) between given values \( u_1 \) and \( u_2 \) at the nodes.

Figure 31.3 shows the shape function along with the corresponding interpolation functions. Notice that the sum of the interpolation functions is equal to one.

In addition, the fact that we are dealing with linear equations facilitates operations such as differentiation and integration. Such manipulations will be important in later sections. The derivative of Eq. (31.2) is

\[
\frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2
\]  

(31.5)

According to Eqs. (31.3) and (31.4), the derivatives of the \( N \)'s can be calculated as

\[
\frac{dN_1}{dx} = -\frac{1}{x_2 - x_1}, \quad \frac{dN_2}{dx} = \frac{1}{x_2 - x_1}
\]  

(31.6)

and, therefore, the derivative of \( u \) is

\[
\frac{du}{dx} = \frac{1}{x_2 - x_1} (-u_1 + u_2)
\]  

(31.7)

In other words, it is a divided difference representing the slope of the straight line connecting the nodes.

The integral can be expressed as

\[
\int_{x_1}^{x_2} u \, dx = \int_{x_1}^{x_2} N_1 u_1 + N_2 u_2 \, dx
\]

Each term on the right-hand side is merely the integral of a right triangle with base \( x_2 - x_1 \) and height \( u \). That is,

\[
\int_{x_1}^{x_2} N_1 \, dx = \frac{1}{2} (x_2 - x_1) u
\]

\[
\int_{x_1}^{x_2} N_2 \, dx = \frac{1}{2} (x_2 - x_1) u
\]

Thus, the entire integral is

\[
\int_{x_1}^{x_2} u \, dx = \frac{u_1 + u_2}{2} (x_2 - x_1)
\]  

(31.8)

In other words, it is simply the trapezoidal rule.

**Obtaining an Optimal Fit of the Function to the Solution.** Once the interpolation function is chosen, the equation governing the behavior of the element must be developed. This equation represents a fit of the function to the solution of the underlying differential equation. Several methods are available for this purpose. Among the most common are the direct approach, the method of weighted residuals, and the variational approach. The outcome of all of these methods is analogous to curve fitting. However, instead of fitting functions to data, these methods specify relationships between the unknowns in Eq. (31.2) that satisfy the underlying PDE in an optimal fashion.
Mathematically, the resulting element equations will often consist of a set of linear algebraic equations that can be expressed in matrix form,

\[ [k][u] = \{F\} \]  \hspace{1cm} (31.9)

where \([k]\) = an element property or stiffness matrix, \([u]\) = a column vector of unknowns at the nodes, and \(\{F\}\) = a column vector reflecting the effect of any external influences applied at the nodes. Note that, in some cases, the equations can be nonlinear. However, for the elementary examples described herein, and for many practical problems, the systems are linear.

### 31.1.3 Assembly

After the individual element equations are derived, they must be linked together or assembled to characterize the unified behavior of the entire system. The assembly process is governed by the concept of continuity. That is, the solutions for contiguous elements are matched so that the unknown values (and sometimes the derivatives) at their common nodes are equivalent. Thus, the total solution will be continuous.

When all the individual versions of Eq. (31.9) are finally assembled, the entire system is expressed in matrix form as

\[ [K][u'] = \{F'\} \]  \hspace{1cm} (31.10)

where \([K]\) = the assemblage property matrix and \([u']\) and \(\{F'\}\) = column vectors for unknowns and external forces that are marked with primes to denote that they are an assembly of the vectors \([u]\) and \(\{F\}\) from the individual elements.

### 31.1.4 Boundary Conditions

Before Eq. (31.10) can be solved, it must be modified to account for the system’s boundary conditions. These adjustments result in

\[ [\tilde{k}][u'] = \{\tilde{F}'\} \]  \hspace{1cm} (31.11)

where the overbars signify that the boundary conditions have been incorporated.

### 31.1.5 Solution

Solutions of Eq. (31.11) can be obtained with techniques described previously in Part Three, such as LU decomposition. In many cases, the elements can be configured so that the resulting equations are banded. Thus, the highly efficient solution schemes available for such systems can be employed.

### 31.1.6 Postprocessing

Upon obtaining a solution, it can be displayed in tabular form or graphically. In addition, secondary variables can be determined and displayed.

Although the preceding steps are very general, they are common to most implementations of the finite-element approach. In the following section, we illustrate how they can be applied to obtain numerical results for a simple physical system—a heated rod.
31.2 FINITE-ELEMENT APPLICATION IN ONE DIMENSION

Figure 31.4 shows a system that can be modeled by a one-dimensional form of Poisson's equation

\[ \frac{d^2 T}{dx^2} = -f(x) \]  \hspace{1cm} (31.12)

where \( f(x) \) = a function defining a heat source along the rod and where the ends of the rod are held at fixed temperatures,

\[ T(0, t) = T_1 \]

and

\[ T(L, t) = T_2 \]

Notice that this is not a partial differential equation but rather is a boundary-value ODE. This simple model is used because it will allow us to introduce the finite-element approach without some of the complications involved in, for example, a two-dimensional PDE.

**Example 31.1**

**Problem Statement.** Solve Eq. (31.12) for a 10-cm rod with boundary conditions of \( T(0, t) = 40 \) and \( T(10, t) = 200 \) and a uniform heat source of \( f(x) = 10 \).

**Solution.** The equation to be solved is

\[ \frac{d^2 T}{dx^2} = -10 \]
Assume a solution of the form

\[ T = ax^2 + bx + c \]

which can be differentiated twice to give \( T'' = 2a \). Substituting this result into the differential equation gives \( a = -5 \). The boundary conditions can be used to evaluate the remaining coefficients. For the first condition at \( x = 0 \),

\[ 40 = -5(0)^2 + b(0) + c \]

or \( c = 40 \). Similarly, for the second condition,

\[ 200 = -5(10)^2 + b(10) + 40 \]

which can be solved for \( b = 66 \). Therefore, the final solution is

\[ T = -5x^2 + 66x + 40 \]

The results are plotted in Fig. 31.5.

### 31.2.1 Discretization

A simple configuration to model the system is a series of equal-length elements (Fig. 31.4b). Thus, the system is treated as four equal-length elements and five nodes.
31.2.2 Element Equations

An individual element is shown in Fig. 31.6a. The distribution of temperature for the element can be represented by the approximation function

\[ \bar{T} = N_1 T_1 + N_2 T_2 \]  

(31.13)

where \( N_1 \) and \( N_2 \) are linear interpolation functions specified by Eqs. (31.3) and (31.4), respectively. Thus, as depicted in Fig. 31.6b, the approximation function amounts to a linear interpolation between the two nodal temperatures.

As noted in Sec. 31.1, there are a variety of approaches for developing the element equation. In this section, we employ two of these. First, a direct approach will be used for the simple case where \( f(x) = 0 \). Then, because of its general applicability in engineering, we will devote most of the section to the method of weighted residuals.

The Direct Approach. For the case where \( f(x) = 0 \), a direct method can be employed to generate the element equations. The relationship between heat flux and temperature gradient can be represented by Fourier’s law:

\[ q = -k' \frac{dT}{dx} \]

where \( q \) = flux [cal/(cm² · s)] and \( k' \) = the coefficient of thermal conductivity [cal/(s · cm · °C)]. If a linear approximation function is used to characterize the element’s temperature, the heat flow into the element through node 1 can be represented by

\[ q_1 = k' \frac{T_1 - T_2}{x_2 - x_1} \]

where \( q_1 \) is heat flux at node 1. Similarly, for node 2,

\[ q_2 = k' \frac{T_2 - T_1}{x_2 - x_1} \]

These two equations express the relationship of the element’s internal temperature distribution (as reflected by the nodal temperatures) to the heat flux at its ends. As such, they constitute our desired element equations. They can be simplified further by recognizing that Fourier’s law can be used to couch the end fluxes themselves in terms of the temperature gradients at the boundaries. That is,

\[ q_1 = -k' \frac{dT(x_1)}{dx} \quad q_2 = k' \frac{dT(x_2)}{dx} \]

which can be substituted into the element equations to give

\[ \frac{1}{x_2 - x_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -\frac{dT(x_1)}{dx} \\ \frac{dT(x_2)}{dx} \end{bmatrix} \]

(31.14)

Notice that Eq. (31.14) has been cast in the format of Eq. (31.9). Thus, we have succeeded in generating a matrix equation that describes the behavior of a typical element in our system.
The direct approach has great intuitive appeal. Additionally, in areas such as mechanics, it can be employed to solve meaningful problems. However, in other contexts, it is often difficult or impossible to derive finite-element equations directly. Consequently, as described next, more general mathematical techniques are available.

The Method of Weighted Residuals. The differential equation (31.12) can be reexpressed as

\[ 0 = \frac{d^2 T}{dx^2} + f(x) \]

The approximate solution [Eq. (31.13)] can be substituted into this equation. Because Eq. (31.13) is not the exact solution, the left side of the resulting equation will not be zero but will equal a residual,

\[ R = \frac{d^2 \tilde{T}}{dx^2} + f(x) \quad (31.15) \]

The method of weighted residuals (MWR) consists of finding a minimum for the residual according to the general formula

\[ \int_D RW_i \, dD = 0 \quad i = 1, 2, \ldots, m \quad (31.16) \]

where \( D \) = the solution domain and the \( W_i \) = linearly independent weighting functions.

At this point, there are a variety of choices that could be made for the weighting function (Box 31.1). The most common approach for the finite-element method is to employ the interpolation functions \( N_i \) as the weighting functions. When these are substituted into Eq. (31.16), the result is referred to as Galerkin’s method,

\[ \int_D RN_i \, dD = 0 \quad i = 1, 2, \ldots, m \]

For our one-dimensional rod, Eq. (31.15) can be substituted into this formulation to give

\[ \int_{x_1}^{x_2} \left[ \frac{d^2 \tilde{T}}{dx^2} + f(x) \right] N_i \, dx \quad i = 1, 2 \]

which can be reexpressed as

\[ \int_{x_1}^{x_2} \frac{d^2 \tilde{T}}{dx^2} N_i(x) \, dx = - \int_{x_1}^{x_2} f(x) N_i(x) \, dx \quad i = 1, 2 \quad (31.17) \]

At this point, a number of mathematical manipulations will be applied to simplify and evaluate Eq. (31.17). Among the most important is the simplification of the left-hand side using integration by parts. Recall from calculus that this operation can be expressed generally as

\[ \int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du \]
Box 31.1 Alternative Residual Schemes for the MWR

Several choices can be made for the weighting functions of Eq. (31.16). Each represents an alternative approach for the MWR. In the **collocation approach**, we choose as many locations as there are unknown coefficients. Then, the coefficients are adjusted until the residual vanishes at each of these locations. Consequently, the approximating function will yield perfect results at the chosen locations but will have a nonzero residual elsewhere. Thus, it is akin to the interpolation methods in Chap. 18. Note that collocation amounts to using the weighting function

\[ W = \delta(x - x_i) \quad \text{for} \quad i = 1, 2, \ldots, n \]

where \( n \) is the number of unknown coefficients and \( \delta(x - x_i) \) is the Dirac delta function that vanishes everywhere but at \( x = x_i \), where it equals 1.

In the **subdomain method**, the interval is divided into as many segments, or “subdomains,” as there are unknown coefficients. Then, the coefficients are adjusted until the average value of the residual is zero in each subdomain. Thus, for each subdomain, the weighting function is equal to 1 and Eq. (31.16) is

\[ \int_{x_{i-1}}^{x_i} R \ dx = 0 \quad \text{for} \quad i = 1, 2, \ldots, n \]

where \( x_{i-1} \) and \( x_i \) are the bounds of the subdomain.

For the least-squares case, the coefficients are adjusted so as to minimize the integral of the square of the residual. Thus, the weighting functions are

\[ W_i = \frac{\partial R}{\partial a_i} \]

which can be substituted into Eq. (31.16) to give

\[ \int_D R \frac{\partial R}{\partial a_i} \ dD = 0 \quad i = 1, 2, \ldots, n \]

or

\[ \frac{\partial}{\partial a_i} \int_D R^2 \ dD = 0 \quad i = 1, 2, \ldots, n \]

Comparison of the formulation with those of Chap. 17 shows that this is the continuous form of regression.

Galerkin’s method employs the interpolation functions \( N_i \) as weighting functions. Recall that these functions always sum to 1 at any position in an element. For many problem contexts, Galerkin’s method yields the same results as are obtained by variational methods. Consequently, it is the most commonly employed version of MWR used in finite-element analysis.

If \( u \) and \( v \) are chosen properly, the new integral on the right-hand side will be easier to evaluate than the original one on the left-hand side. This can be done for the term on the left-hand side of Eq. (31.17) by choosing \( N_i(x) \) as \( u \) and \( (d^2 T / dx^2) \) as \( dv \) to yield

\[ \int_{x_1}^{x_2} N_i(x) \frac{d^2 T}{dx^2} \ dx = N_i(x_2) \frac{d T}{dx} \bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d T}{dx} \ \frac{d N_i}{dx} \ dx \quad i = 1, 2 \quad (31.18) \]

Thus, we have taken the significant step of lowering the highest-order term in the formulation from a second to a first derivative.

Next, we can evaluate the individual terms that we have created in Eq. (31.18). For \( i = 1 \), the first term on the right-hand side of Eq. (31.18) can be evaluated as

\[ \frac{d N_1}{dx} \bigg|_{x_1}^{x_2} = N_1(x_2) \frac{d T(x_2)}{dx} - N_1(x_1) \frac{d T(x_1)}{dx} \]

However, recall from Fig. 31.3 that \( N_1(x_2) = 0 \) and \( N_1(x_1) = 1 \), and therefore,

\[ \frac{d N_1}{dx} \bigg|_{x_1}^{x_2} = \frac{d T(x_1)}{dx} \quad (31.19) \]
Similarly, for \( i = 2 \),

\[
N_2(x) \left. \frac{d \tilde{T}}{dx} \right|_{x_1}^{x_2} = \frac{d \tilde{T}(x_2)}{dx} - \frac{d \tilde{T}(x_1)}{dx}
\]  

(31.20)

Thus, the first term on the right-hand side of Eq. (31.18) represents the natural boundary conditions at the ends of the elements.

Now, before proceeding let us regroup by substituting our results back into the original equation. Substituting Eqs. (31.18) through (31.20) into Eq. (31.17) and rearranging gives for \( i = 1 \),

\[
\int_{x_1}^{x_2} \frac{d \tilde{T}}{dx} \frac{d N_1}{dx} \, dx = -\frac{d \tilde{T}(x_1)}{dx} + \int_{x_1}^{x_2} f(x) N_1(x) \, dx
\]

(31.21)

and for \( i = 2 \),

\[
\int_{x_1}^{x_2} \frac{d \tilde{T}}{dx} \frac{d N_2}{dx} \, dx = \frac{d \tilde{T}(x_2)}{dx} + \int_{x_1}^{x_2} f(x) N_2(x) \, dx
\]

(31.22)

Notice that the integration by parts has led to two important outcomes. First, it has incorporated the boundary conditions directly into the element equations. Second, it has lowered the highest-order evaluation from a second to a first derivative. This latter outcome yields the significant result that the approximation functions need to preserve continuity of value but not slope at the nodes.

Also notice that we can now begin to ascribe some physical significance to the individual terms we have derived. On the right-hand side of each equation, the first term represents one of the element’s boundary conditions and the second is the effect of the system’s forcing function—in the present case, the heat source \( f(x) \). As will now become evident, the left-hand side embodies the internal mechanisms that govern the element’s temperature distribution. That is, in terms of the finite-element method, the left-hand side will become the element property matrix.

To see this, let us concentrate on the terms on the left-hand side. For \( i = 1 \), the term is

\[
\int_{x_1}^{x_2} \frac{d \tilde{T}}{dx} \frac{d N_1}{dx} \, dx
\]

(31.23)

Recall from Sec. 31.1.2 that the linear nature of the shape function makes differentiation and integration simple. Substituting Eqs. (31.6) and (31.7) into Eq. (31.23) gives

\[
\int_{x_1}^{x_2} \frac{T_1 - T_2}{(x_2 - x_1)^2} \, dx = \frac{1}{x_2 - x_1}(T_1 - T_2)
\]

(31.24)

Similar substitutions for \( i = 2 \) [Eq. (31.22)] yield

\[
\int_{x_1}^{x_2} \frac{-T_1 + T_2}{(x_2 - x_1)^2} \, dx = \frac{1}{x_2 - x_1}(-T_1 + T_2)
\]

(31.25)

Comparison with Eq. (31.14) shows that these are similar to the relationships that were developed with the direct method using Fourier’s law. This can be made even clearer by
reexpressing Eqs. (31.24) and (31.25) in matrix form as

\[
\frac{1}{x_2 - x_1} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2
\end{bmatrix}
\]

Substituting this result into Eqs. (31.21) and (31.22) and expressing the result in matrix form gives the final version of the element equations

\[
\frac{1}{x_2 - x_1} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}\{T\} = \begin{bmatrix}
\frac{dT(x_1)}{dx} \\
\frac{dT(x_2)}{dx}
\end{bmatrix} + \begin{bmatrix}
\int_{x_1}^{x_2} f(x)N_1(x) \, dx \\
\int_{x_1}^{x_2} f(x)N_2(x) \, dx
\end{bmatrix}
\]

(31.26)

Element stiffness matrix

Boundary condition

External effects

Note that aside from the direct and the weighted residual methods, the element equations can also be derived using variational calculus (for example, see Allaire, 1985). For the present case, this approach yields equations that are identical to those derived above.

### Example 31.2

**Element Equation for a Heated Rod**

**Problem Statement.** Employ Eq. (31.26) to develop the element equations for a 10-cm rod with boundary conditions of \( T(0, t) = 40 \) and \( T(10, t) = 200 \) and a uniform heat source of \( f(x) = 10 \). Employ four equal-size elements of length = 2.5 cm.

**Solution.** The heat source term in the first row of Eq. (31.26) can be evaluated by substituting Eq. (31.3) and integrating to give

\[
\int_{0}^{2.5} 10 \cdot \frac{2.5 - x}{2.5} \, dx = 12.5
\]

Similarly, Eq. (31.4) can be substituted into the heat source term of the second row of Eq. (31.26), which can also be integrated to yield

\[
\int_{0}^{2.5} 10 \cdot \frac{x - 0}{2.5} \, dx = 12.5
\]

These results along with the other parameter values can be substituted into Eq. (31.26) to give

\[
0.4T_1 - 0.4T_2 = \frac{dT}{dx}(x_1) + 12.5
\]

and

\[
-0.4T_1 + 0.4T_2 = \frac{dT}{dx}(x_2) + 12.5
\]

### 31.2.3 Assembly

Before the element equations are assembled, a global numbering scheme must be established to specify the system’s topology or spatial layout. As in Table 31.1, this defines the